Lecture 9. Trading Strategies Involving Options II
Binomial Trees

EF4321. Derivatives and Risk Management

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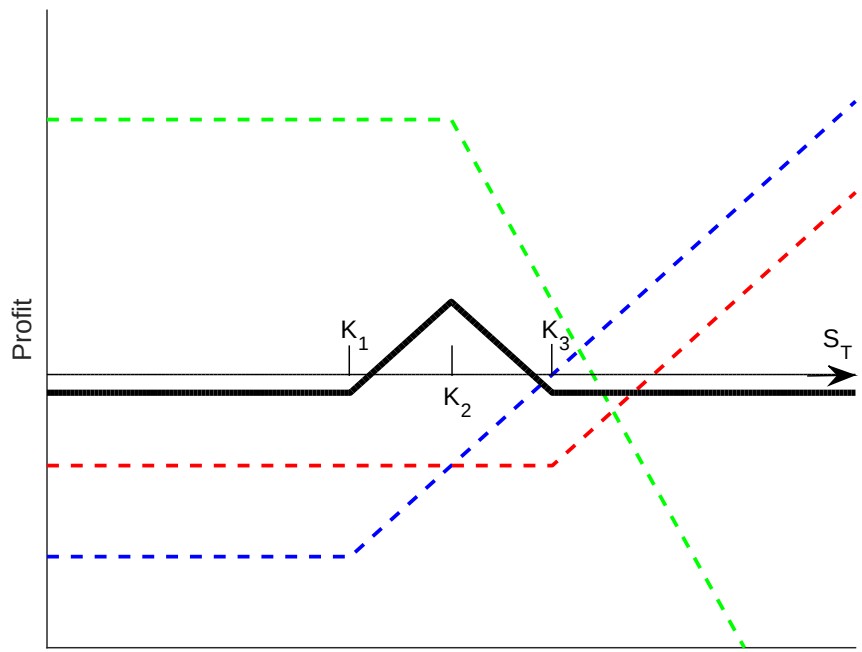
Lecture Outline

- Trading Strategies Involving Options II
  - Spreads
  - Combinations
- Binomial Trees
  - One-step binomial model
  - Risk-Neutral Valuation

Spreads - Butterfly Spreads

- Butterfly spread is a strategy where profit is maximized at a certain stock price. This is appropriate for investors who believe that large stock price moves are unlikely.

- To create, we buy a European call with strike price of K_1, buy a European call with K_3, and sell two European calls with K_2, where $K_1 < K_2 < K_3$. All options have the same expiration dates.



Spreads - Butterfly Spreads

- Payoff at expiration

Action	Payoff at time T			
	$S_T \geq K_3$	$K_3 > S_T \geq K_2$	$K_2 > S_T \geq K_1$	$K_1 > S_T$
long call with K_1	$S_T - K_1$	$S_T - K_1$	$S_T - K_1$	0
long call with K_3	$S_T - K_3$	0	0	0
short 2 calls with K_2	$-2(S_T - K_2)$	$-2(S_T - K_2)$	0	0
net payoff	$2K_2 - K_1 - K_3$	$-S_T + 2K_2 - K_1$	$S_T - K_1$	0

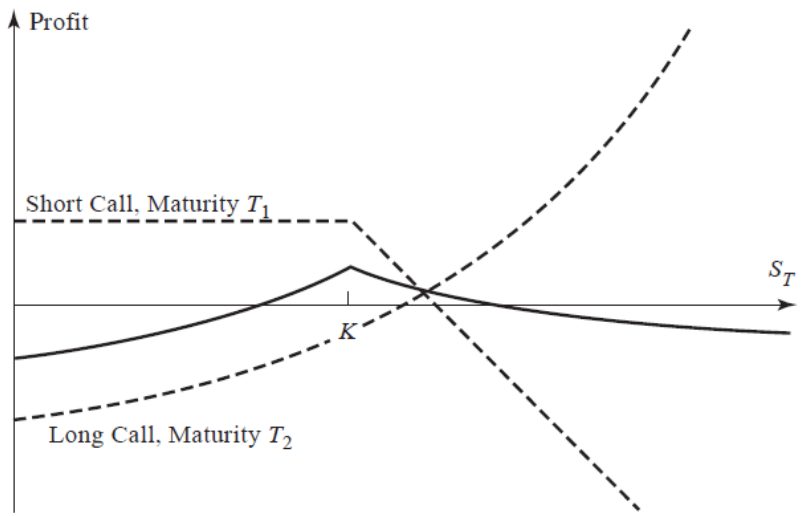
- The butterfly spreads can also be created using puts (buying a put with K_1, buying a put with K_3 and selling two puts with K_2.)

Spreads - Calendar Spreads

- Calendar spreads consist of European options with the same strike price K and different expiration dates.

- To create, we sell a European call with maturity T_1 and buy a European call with maturity $T_2 (> T_1)$.

Figure 11.8 Profit from calendar spread created using two call options, calculated at the time when the short-maturity call option expires.



Spreads - Calendar Spreads

- At T_1, T_1-call is expired. Also, we sell T_2-call in the market. The resulting payoff at T_1 is

$$-\max(S_{T_1} - K, 0) + c_{T_1}(T_2)$$

  where $c_{T_1}(T_2)$ is the price of T_2-call at T_1.

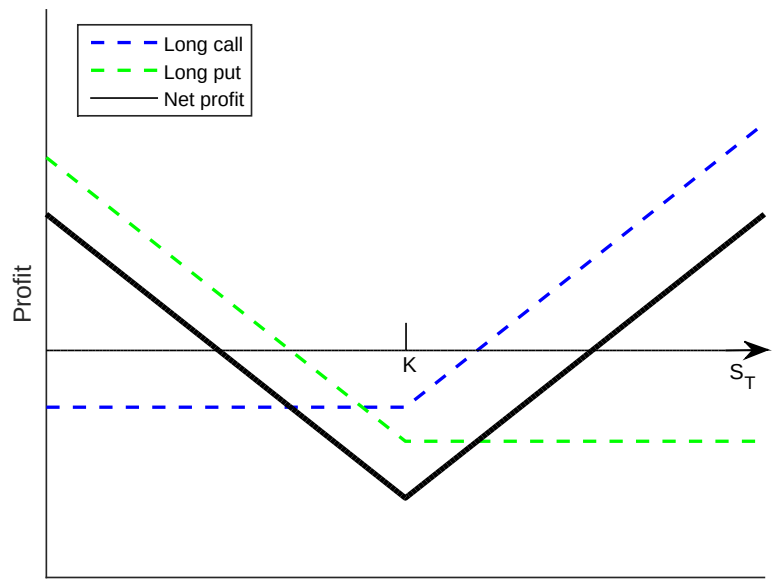
- Usually, a longer-maturity call option has a higher price, so this strategy requires an initial investment.

- The investor makes profit when stock price is close to K. This pattern is similar to the butterfly spreads.

Combinations - Straddle

- A combination is a trading strategy that involves taking a position in both calls and puts.

- Straddle
  - This consists of buying a European call and a European put with the same strike price and expiration date.



  - This is appropriate for an investor who expects a large move in stock price but is uncertain about a direction.

Combinations - Straddle

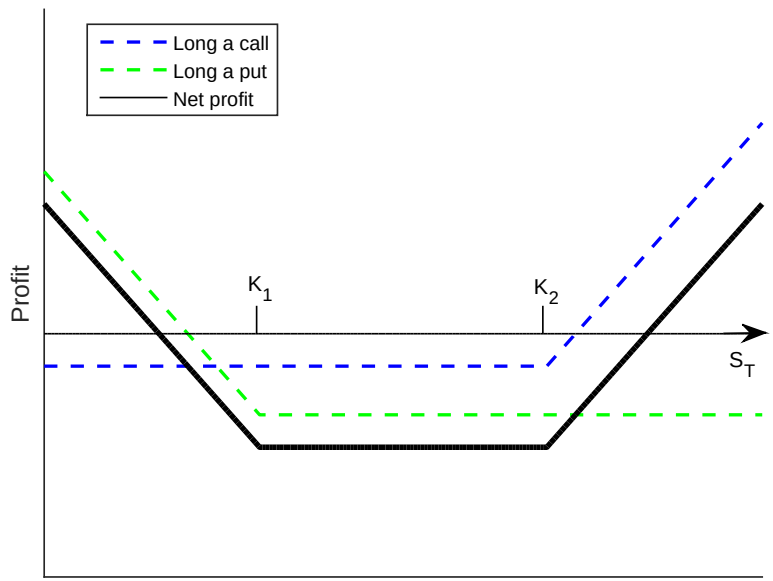
- Payoff at maturity

Action	Payoff at time T	
	$S_T \geq K$	$K > S_T$
long call	$S_T - K$	0
long put	0	$K - S_T$
Net	$S_T - K$	$K - S_T$

- Initially, an investor needs to pay option prices for both call and put.

Combinations - Strangle

- A disadvantage of straddle is that it requires a large initial investment to buy both call and put.
- Strangle is similar to straddle, but we buy cheaper call and put.
- In particular, we buy a European put with K_1 and a European call with $K_2 (> K_1)$.



Combinations - Strangle

- Payoff at maturity

Action	Payoff at time T		
	$S_T \geq K_2$	$K_2 > S_T \geq K_1$	$K_1 > S_T$
long call with K_2	$S_T - K_2$	0	0
long put with K_1	0	0	$K_1 - S_T$
Net	$S_T - K_2$	0	$K_1 - S_T$

- The profit structure is similar to straddle. However, the downside risk when the stock price does not move much is less than straddle.

Binomial Trees

Determining Option Prices

- We have characterized option prices using lower/upper bounds and the put-call parity. Still, we don't yet have a tool to determine the exact price of an option.

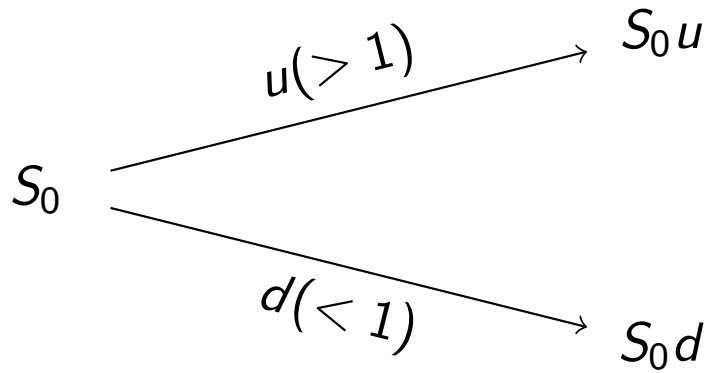
- To find the exact price of an option, we need a model describing how the underlying stock price will move in the future.

- Models for pricing options
  1. Binomial model
  2. Black-Scholes-Merton model

Binomial Model - Setting

- Assumptions
  - Stock price follows a random walk.

    : In one time step, the stock price can move up or down by a certain amount (only two possible paths).



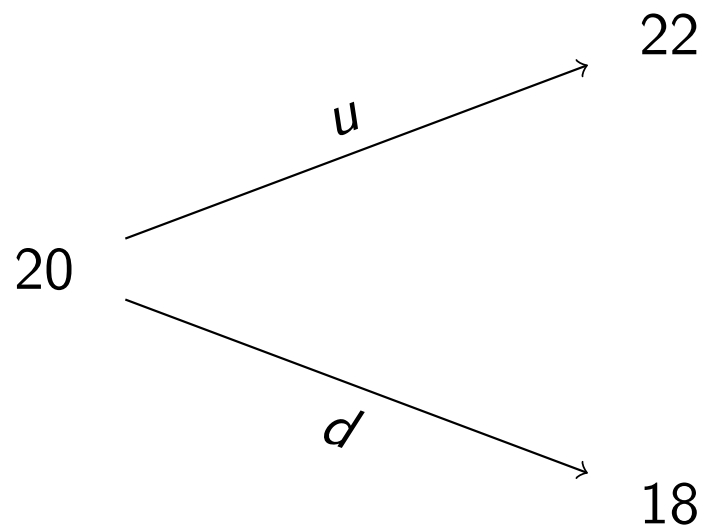
  - There is no arbitrage.

- This may look too simplistic to reflect the reality. Later, we will extend the model to allow multiple steps until the option expiration.

One-Step Binomial Model

e.g. A 3-month European call option has strike price \$21. The risk-free interest rate is 12% per annum.

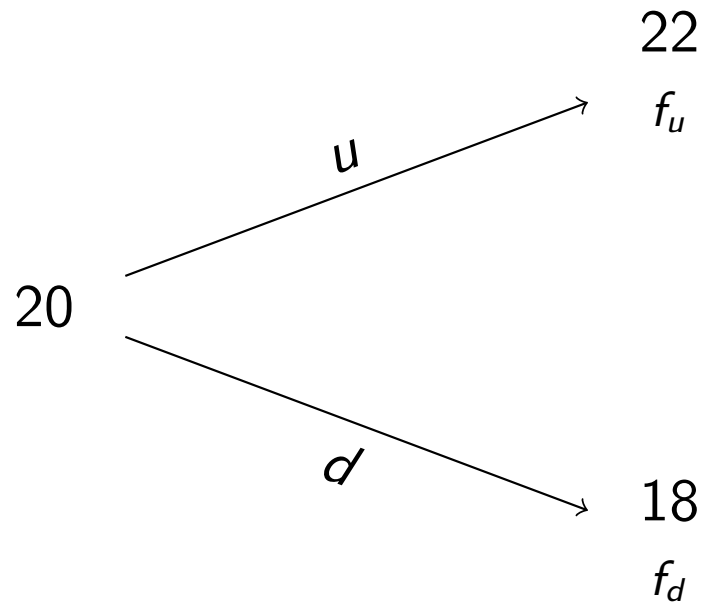
- Current stock price is \$20. The price can move either up to \$22 or down to \$18 during the life of the option.



- What is the price of the call option?

One-Step Binomial Model

1. To price the call, we first determine option payoffs at T:



- f_u(option value when stock price is up)$= \max(22 - K, 0) = 1$
- f_d(option value when stock price is down) $= \max(18 - K, 0) = 0$

One-Step Binomial Model

2. Now, we suppose that we stand in time 0. Find a portfolio that replicates the option payoff in every case at T (using stock and bond):

   - Let x denote the number of shares and y the face value of the bond (in dollar) in the replicating portfolio. We want x and y such that

$$\begin{cases} 22x + y = 1 & (\text{at } u) \\ 18x + y = 0 & (\text{at } d) \end{cases}$$

   - Solving for the unknowns gives $x = 0.25$, $y = -4.5$. Thus, the replicating portfolio consists of buying 0.25 shares and selling a bond with the face value of -\$4.5.

One-Step Binomial Model

3. The option price at time 0 should be equal to the price of the replicating portfolio. Otherwise, an arbitrage exists.

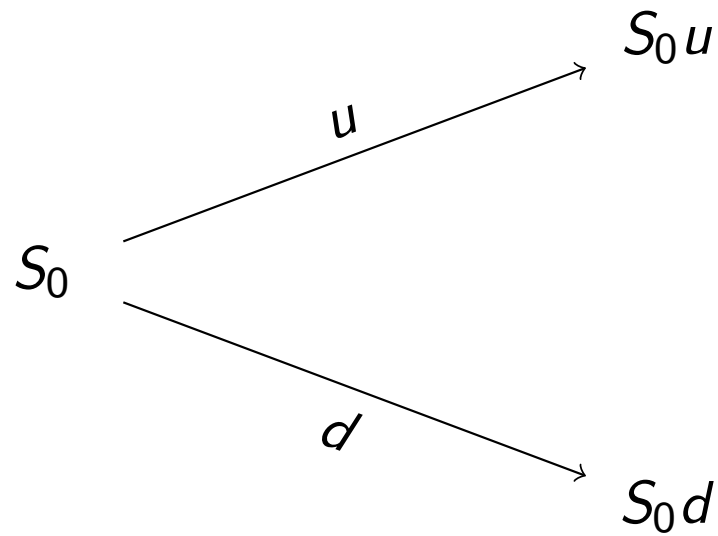
   - The price of the replicating portfolio is

   $$S_0 x + y e^{-rT} = 20 \times 0.25 - 4.5 e^{-0.12 \times 3/12} = 0.633$$

   - Hence, the option price is \$0.633.

One-Step Binomial Model - General case

- A European call option has strike price K and expiration date T. The risk-free interest rate is r per annum.
- Current stock price is S_0. The price can move either up by u or down by d during the life of the option.



- Notice that we do not assign any probability for up/down movement.

One-Step Binomial Model - General case

1. Find option payoffs at T: $\begin{cases} f_u &= \max(S_0 u - K, 0) \\ f_d &= \max(S_0 d - K, 0) \end{cases}$

2. Find the replicating portfolio (x: number of shares, y: face value of bond).

   - We want to find x and y such that

$$\begin{cases} (S_0 u)x + y = f_u & (\text{at } u) \\ (S_0 d)x + y = f_d & (\text{at } d) \end{cases}$$

   - Solving for the unknowns, we obtain

$$x = \frac{f_u - f_d}{S_0 u - S_0 d}, \quad y = \frac{u f_d - d f_u}{u - d}$$

One-Step Binomial Model - General case

3. Calculate the present value (at time 0) of the replicating portfolio.

$$\begin{aligned} S_0 x + y e^{-rT} &= \frac{f_u - f_d}{u-d} + e^{-rT}\frac{uf_d - df_u}{u-d} \\ &= e^{-rT}\left[\frac{e^{rT}f_u - e^{rT}f_d}{u-d} + \frac{uf_d - df_u}{u-d}\right] \\ &= e^{-rT}\left[\frac{e^{rT}-d}{u-d}f_u + \frac{u-e^{rT}}{u-d}f_d\right] \\ &= e^{-rT}\left[p \times f_u + (1-p)\times f_d\right] \end{aligned}$$

where $p = \frac{e^{rT}-d}{u-d}$.

One-Step Binomial Model

e.g. Go back to the previous example of the European call option with $K = 21$, and $T = 3$ months. The risk-free interest rate is 12% per annum. The current stock price is 20. The price can either move up to 22 or down to 18 during the life of the option.

- We priced the option using the replicating portfolio. Alternatively, we can use the option pricing formula. Here, $u = 22/20 = 1.1$ and $d = 18/20 = 0.9$.

- Then,

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 1/4} - 0.9}{1.1 - 0.9} = 0.652$$

- The price of the call option is

$$\begin{aligned} e^{-rT}\left[pf_u + (1-p)f_d\right] &= e^{-0.12 \times 1/4}\left[0.652 \times 1 + (1 - 0.652) \times 0\right] \\ &= \$0.633 \end{aligned}$$

Interpretation: Risk-Neutral Valuation

- The option price $e^{-rT}\left[pf_u + (1-p)f_d\right]$ is similar to the form we would obtain from DCF, when p is interpreted as a probability.

- However, the form is not exactly the same as the DCF.
  - Recall that in DCF, a riskier cash flows is discounted at a higher rate, say r_{call} (e.g. CAPM).
  - However, in the result of option price, the risky option payoff is discounted at the risk-free interest rate.

- Risk-Neutral Valuation
  - The discount rate in the option price is determined **as if** investors do not require a higher return for a riskier investment, that is, as if they are **risk-neutral**.

Risk-Neutral Valuation vs. DCF

- We call p the risk-neutral probability of an increase in stock price.
- p is different from the real probability we observe in data. To distinguish, let p^* denote the real probability of an increase in the stock price.
- Risk-neutral valuation vs. DCF

	Option Pricing (Risk-Neutral Valuation)	DCF
Discount rate	r	r_{call}
Probability	p	p^*
Option price	$e^{-rT}\left[pf_u + (1-p)f_d\right]$	$e^{-r_{\text{call}}T}\left[p^* f_u + (1-p^*)f_d\right]$

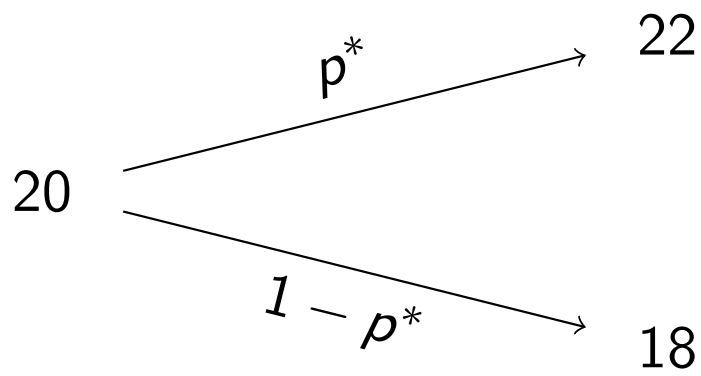
Pricing Options Using DCF

- Instead of risk-neutral valuation, we can price an option using the discounted cash flow (DCF) approach.
- To price an option using DCF, we need to find the real probability p^* and the discount rate r_{call} for the option.
- First, to find p^*, we suppose that risk-averse investors require the stock return to be α per annum in the real world.
- Then, we solve for p^* by setting α equal to the expected return in the binomial tree.

$$p^* = \frac{e^{\alpha T} - d}{u - d}$$

Pricing Options Using DCF

Q1. Revisit the previous example, where stock price in 3 months is given as below. In the real world, risk-averse investors require the return on stock to be 16% per annum. What is the real probability p^* of an increase in the stock price?



Pricing Options Using DCF

- In pricing using DCF, the next step is to find the discount rate r_{call}.
- To determine the discount rate, we use the fact that a portfolio of stock and bond replicates the call option in the binomial tree.
- Then, the required return on option should be equal to the weighted average of stock return (α) and bond return (r).
- The weight is determined by the fraction of stock and bond components in the portfolio.

$$\begin{cases} \text{weight on stock: } \dfrac{S_0 x}{S_0 x + y e^{-rT}} \\ \text{weight on bond: } \dfrac{y e^{-rT}}{S_0 x + y e^{-rT}} \end{cases}$$

Pricing Options Using DCF

Q2. In the previous tree, consider a 3-month call option with the strike price of 21. The risk-free rate is 12% per annum. The required return on stock is 16% per annum. What is the discount rate r_{call} for the call in DCF?

Pricing Options Using DCF

Q2. In the previous tree, consider a 3-month call option with the strike price of 21. The risk-free rate is 12% per annum. The required return on stock is 16% per annum. What is the discount rate r_{call} for the call in DCF?

Answer: r_{call} is the weighted average of stock return and bond return, where the weight is the fraction of an asset in the entire portfolio value.

$$\begin{aligned} e^{r_{\text{call}} \times 3/12} &= \frac{xS_0}{S_0 x + ye^{-rT}} \times e^{\alpha \times 3/12} + \frac{ye^{-rT}}{S_0 x + ye^{-rT}} \times e^{r \times 3/12} \\ &= \frac{(20)(0.25)}{0.633} \times e^{0.16/4} + \frac{-4.5e^{-0.12 \times 1/4}}{0.633} \times e^{0.12/4} \\ &= 1.112258 \end{aligned}$$

Thus, the discount rate for the call is 42.56% per annum.

Pricing Options Using DCF

Q3. Calculate the option price in Q2 using DCF. Is the price the same as the price from the risk-neutral valuation?

Risk-Neutral Valuation vs. DCF

- The option price from the risk-neutral valuation is **the same** as the price from the DCF.
- If required return on stock α is higher than the risk-free rate r, it follows that $p < p^*$.
- It implies that in the risk-neutral valuation, we amplify the probability of a bad outcome for stock investors.
- We interpret this as the probability being modified to incorporate investors' risk-aversion.

Risk-Neutral Valuation

- Risk-neutral valuation is an **interpretation** of the option pricing formula obtained from the replicating portfolio.

- This interpretation is also useful for multi-step binomial models in the next week.

Things To Do

- Read the textbook chapters 12.3 - 12.4, 13.1 - 13.2
- Assignment 6 (due on 12 November at 11 pm)